

# The $GL(4, R)$ Yang-Mills Theory of Gravity Predicts An Inflationary Scenario For The Evolution of The Primordial Universe

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## Abstract

We show that the  $GL(4, R)$  gauge theory of gravity will admit an exponentially expanding solution for the primordial evolution of the Universe. This inflationary phase of the Universe is driven by a rapidly changing  $GL(4, R)$  gauge field configuration. No cosmological constant and no rolling scalar field is required. This inflationary expansion will then be shown to slow down to a much slower expansion rate during the late time evolution of the Universe.

Keywords: gravitation, Yang-Mills, Einstein, cosmology, inflation

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## I. INTRODUCTION

Recently we have proposed a vector theory of gravity [1]. In this theory the four dimensional Spacetime is assumed to be equipped with a non-dynamical metric  $g_{\mu\nu}$  which is the background metric to give the Spacetime the notion of world length. Any given metric is legitimate, but only those metrics that extremize our proposed action are the metrics that will be observed classically.

Our proposed action comes from our observation that some fundamental laws of Nature, namely the Law of Inertia and the Causality Principle, are preserved under a family of transformation of reference frames called the affine transformations  $GL(4, R)$ , on every local patch of the Spacetime. Therefore we believe that it is natural for us to assume that physics should be invariant under these local transformations.

These local transformations form a Lie group, and hence we are tempted to construct a local Yang-Mills theory [1] based on this local group. This Yang-Mills theory is, of course, a vector theory, and has 16 gauge vector boson  $A_{n\mu}^m$ . Two sets of equations will follow when we vary the metric  $g_{\mu\nu}$  and the gauge potentials  $A_{n\mu}^m$  independently so as to extremize the Yang-Mills action when we are looking for classical solutions to the theory. The first set of equations, which will be called the Stephenson Equation [2], gives algebraic relations among the various components of the Yang-Mills strength tensor. The second set of equations is the Yang-Mills equation in the presence of a background metric, and will be called the Stephenson-Kilmister-Yang Equation [3].

The Yang-Mills action can be transformed into a form that might look familiar to us when we make a variable substitution of the 64 variables  $A_{n\mu}^m$  by a new set of 64 variables  $\Gamma_{\tau\mu}^\rho$  through

$$A_{n\mu}^m = e_\rho^m e_n^\tau \Gamma_{\tau\mu}^\rho + e_\tau^m \partial_\mu e_n^\tau, \quad (1)$$

where  $e_\rho^m$  are the vierbein fields for the background metric in reference to a local Minkowskian frame (we shall use the Latin and Greek indices to denote, respectively, local Minkowskian and world coordinates. A hat is always put on an index when we want to emphasize that we are talking about local coordinates). With the new variables  $\Gamma_{\tau\mu}^\rho$ , our proposed action will look like [1]

$$S_{\text{YM}}[g, A, \partial A] = S_{\text{YM}}[g, \Gamma, \partial \Gamma] = \kappa \int \sqrt{-g} d^4x g^{\mu\mu'} g^{\nu\nu'} (F_{n\mu\nu}^m F_{m\mu'\nu'}^n) \quad (2)$$

$$= \kappa \int \sqrt{-g} d^4x g^{\mu\mu'} g^{\nu\nu'} (R^\lambda_{\sigma\mu\nu} R^\sigma_{\lambda\mu'\nu'}).$$

where  $F^m_{n\mu\nu} = \partial_\mu A^m_{n\nu} - \partial_\nu A^m_{n\mu} + A^m_{p\mu} A^p_{n\nu} - A^m_{p\nu} A^p_{n\mu}$  stands for the Yang-Mills field strength tensor,  $R^\lambda_{\sigma\mu\nu} = \partial_\mu \Gamma^\lambda_{\sigma\nu} - \partial_\nu \Gamma^\lambda_{\sigma\mu} + \Gamma^\lambda_{\rho\mu} \Gamma^\rho_{\sigma\nu} - \Gamma^\lambda_{\rho\nu} \Gamma^\rho_{\sigma\mu}$  stands for the Riemannian curvature tensor, and  $\kappa$  is a dimensionless coupling constant for the theory.

The corresponding Stephenson and Stephenson-Kilmister-Yang Eequations will be, respectively

$$\begin{aligned} R^\lambda_{\sigma\theta\rho} R^\sigma_{\lambda\tau}{}^\rho - \frac{1}{4} g_{\theta\tau} R^\lambda_{\sigma}{}^{\xi\rho} R^\sigma_{\lambda\xi\rho} &= \frac{1}{2\kappa} T_{\theta\tau}, \\ \nabla_\rho(\Gamma)(\sqrt{-g} R^\beta{}_{\sigma}{}^{\rho\lambda}) &= \frac{1}{\kappa} \sqrt{-g} S^\beta{}_{\sigma}{}^{\rho\lambda}. \end{aligned} \quad (3)$$

The  $T_{\theta\tau}$  and  $S^\beta{}_{\sigma}{}^{\rho\lambda}$  are respectively the metric energy-momentum tensor and the gauge current tensor of the matter source [1].

Before we will make further discussions, we find it imperative for us to stress that the action given in Eq. 2, though looks very similar to the ones found in many occasions in the discussion of the theory of gravity, is fundamentally different from them because there exists no prior relationship between  $g_{\mu\nu}$  and  $\Gamma^\rho_{\tau\mu}$  in our theory. The geometric objects here, such as the connections and the Riemann curvature tensor, are in fact, the Yang-Mills gauge potentials and the Yang-Mills field strength tensors of  $GL(4, R)$  in disguise [1]. We have converted the original Yang-Mills theory into a theory with geometric languages because we want to make use of the many works done in the past decades by people working in the geometric theory of gravity. The theory here is not a higher derivative theory of the metric and the only dynamical variables are the Yang-Mills potentials which satisfy a second order differential equation.

The theory given in Eqs. 1, 2 and 3 are shown to be able to comply with the gravitational tests in the solar system because the Yang-Mills potentials induce the Schwarzschild metric as one of the admissible metrics [1]. Furthermore there are some other physical phenomena that are predicted by the above theory which are not predicted by the theory of General Relativity. These new predictions include the existence of two gravitational copies of matter which reproduce the effects of Dark Matter as observed in astronomy and also the existence of a primordial torsion which mimics the effects of Dark Energy in the late time evolution of the Universe [1].

## II. COSMOLOGY FROM THE POINT OF VIEW OF THE $GL(4, R)$ YANG-MILLS THEORY

In order to be sure that this vector theory of gravity, basing on the local gauge theory of  $GL(4, R)$  with a non-dynamical world metric, is a viable theory of gravity, we must put this theory on test with the well-known thermal history of the Universe.

In this article, we are going to show that this vector theory of gravity can accommodate an inflationary expansion of the Universe during its early phase of evolution, and this inflationary expansion will then slow down and will take up a much slower accelerating expansion during its late time evolution.

To this end, we will start by searching for solutions in which the Yang-Mills potentials with symmetric gauge indices are vanishing, and that the cosmic metric will be taken as the spatially flat Friedmann-Lematre-Robertson-Walker (FLRW) form

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2). \quad (4)$$

Furthermore we will make some assumptions on the form of our cosmic Yang-Mills potential  $A_{n\mu}^m$ . Because of the fact that the Yang-Mills potentials are related to the connections  $\Gamma_{\tau\mu}^\rho$  through Eq. 1, an assumption on the form of the cosmic Yang-Mills potentials will be equivalent to an assumption on the form of the cosmic connections, which are now compatible with the metric because the Yang-Mills potentials that we are seeking are anti-symmetric in their gauge indices.

There are totally 24 components for the torsion tensor  $\tau_{\alpha\beta}^\rho$ , which are the anti-symmetric parts of the connections. The connections will then have two parts, namely the Christoffel symbols and terms composing of the torsions and the metric.

We choose to describe the cosmic torsion tensor by what are observed by a local Minkowskian observer. This observer will also see 24 local components. These 24 local components will fall into categories in accordance with their parity signatures under the local spatial parity operations (which consist of either one spatial inversion, two spatial inversions or three spatial inversions) of the local Minkowskian frame. Of all these 24 components, only 3 are invariant under the above said spatial parity operations. They are  $\tau_{\hat{0}\hat{1}\hat{1}}$ ,  $\tau_{\hat{0}\hat{2}\hat{2}}$ , and  $\tau_{\hat{0}\hat{3}\hat{3}}$ . We have use a hat for each index because we want to emphasize that these are the components measured by a local Minkowskian observer. We shall then assume that

only the torsion components that are invariant under local parity operations will show up in the cosmic evolution of the Universe.

Local isotropy will also play a role in the determination of the form of the local torsion components too. Local isotropy will require that  $\tau_{\hat{0}\hat{1}\hat{1}}$ ,  $\tau_{\hat{0}\hat{2}\hat{2}}$  and  $\tau_{\hat{0}\hat{3}\hat{3}}$  are all equal. Homogeneity will require that they are functions of time only. Hence we have arrived at the conclusion that we will concern ourselves only with cosmos which have the following local torsion components

$$\tau_{\hat{0}\hat{1}\hat{1}} = \tau_{\hat{0}\hat{2}\hat{2}} = \tau_{\hat{0}\hat{3}\hat{3}} \equiv \frac{\xi(t)}{2}. \quad (5)$$

See also the works of Ramaswamy and Yasskin [4], and Baekler and Hehl [5] and Chen, Hsu and Yeung [6] for the selection of the local torsion components in their discussions of cosmologies in the Poincare Gauge Theory of Gravity.

With the torsion components given in Eq. 5 and the metric given in Eq.4, we can calculate the connections, and, in turn, the Riemann curvature tensor. We will have the following non-vanishing local components:

$$\begin{aligned} R_{\hat{0}\hat{1}\hat{0}\hat{1}} &\equiv D = \frac{\ddot{a}}{a} - \xi \frac{\dot{a}}{a} - \dot{\xi}, \\ R_{\hat{0}\hat{2}\hat{0}\hat{2}} = R_{\hat{0}\hat{3}\hat{0}\hat{3}} &\equiv E = -\frac{\ddot{a}}{a} + \xi \frac{\dot{a}}{a} + \dot{\xi}, \\ R_{\hat{1}\hat{2}\hat{1}\hat{2}} = R_{\hat{1}\hat{3}\hat{1}\hat{3}} &\equiv M = \left(\frac{\dot{a}}{a} - \xi\right)^2, \\ R_{\hat{2}\hat{3}\hat{2}\hat{3}} &\equiv N = -\left(\frac{\dot{a}}{a} - \xi\right)^2. \end{aligned} \quad (6)$$

Using these local curvature components, we can write the Stephenson and the Stephenson-Kilmister-Yang equations as

$$\begin{aligned} (D^2 + 2E^2 - 2M^2 - N^2) &= \frac{1}{2\kappa} T_{\hat{0}\hat{0}}, \\ (2E^2 + N^2 - D^2 - 2M^2) &= \frac{1}{2\kappa} T_{\hat{1}\hat{1}}, \\ (D^2 - N^2) &= \frac{1}{2\kappa} T_{\hat{2}\hat{2}} = \frac{1}{2\kappa} T_{\hat{3}\hat{3}}, \end{aligned} \quad (7)$$

$$2\left(\frac{\dot{a}}{a}\right)E + \dot{E} + 2\left(\frac{\dot{a}}{a} - \xi\right)M = 0. \quad (8)$$

In Eq. 8 and in the subsequent discussions, we will assume that the cosmic gauge current tensor  $S_{\sigma}^{\beta\lambda}$  of the matter field source will be averaged out during the course of evolution of the Universe.

Note the fact that Eq. 8 is the Yang-Mills Equation for the  $GL(4, R)$  group while Eq. 7 is the algebraic equations for the various components of the Yang-Mills strength tensor.

See also the works of Refs. [4], [5] and Chen, Hsu and Yeung [6] for the discussions of cosmic expansion under different situations.

There is a hidden constraint built in Eq. 7 if we are going to impose  $T_{11} = T_{22} = T_{33}$  because of isotropy requirement. Namely that the matter metric energy-momentum tensor components should satisfy the algebraic relation of

$$T_{11} = T_{22} = T_{33} = \frac{1}{3}T_{00}. \quad (9)$$

If we use  $\rho$  to denote  $T_{00}$  and identify it as the local energy density and use  $p$  to denote  $T_{ii}$ , where  $i = 1, 2$  or  $3$ , and identify it as the pressure, then the above relation is just the equation of state for the matter fields,

$$p = \frac{1}{3}\rho. \quad (10)$$

During the expansion of the Universe, the Universe is supposed to undergo an adiabatic process, and hence satisfies the First Law of Thermodynamics and the equation of state of Eq. 10 can be integrated to give

$$\rho = \frac{6\kappa A^4}{a^4}, \quad (11)$$

where  $6\kappa A$  is an integration constant.

Hence the cosmic equations that we are going to solve are

$$\left(\frac{\ddot{a}}{a} - \xi \frac{\dot{a}}{a}\right)^2 - \left(\frac{\dot{a}}{a} - \xi\right)^4 = \left(\frac{A}{a}\right)^4, \quad (12)$$

$$2\left(\frac{\dot{a}}{a}\right)\left(-\frac{\ddot{a}}{a} + \xi \frac{\dot{a}}{a} + \dot{\xi}\right) + \frac{d}{dt}\left(-\frac{\ddot{a}}{a} + \xi \frac{\dot{a}}{a} + \dot{\xi}\right) + 2\left(\frac{\dot{a}}{a} - \xi\right)^3 = 0. \quad (13)$$

These two equations are far more complicated than the Friedmann Equations. And in the following we shall investigate the implications of these two equations on the evolution of the Universe.

### III. THE EVOLUTION OF THE EARLY UNIVERSE

Now consider the situation in which  $\xi$  is an *extremely small constant*, say  $\xi = 10^{-18}\text{sec}^{-1}$ . And also consider the case in which the initial condition for  $a(t)$  at  $t = 0$  (i.e.  $a(0)$ ) of our

cosmic equation is *an extremely small number*. With an extremely small  $a(t)$ , the matter in the Universe will be highly relativistic and we will have  $p = \frac{1}{3}\rho$  as the equation of state, and hence Eq. 11 will be satisfied. At the moment when  $t \approx 0$ , the right-hand-side of the Eq. 12 will be huge for the reason that  $a(t)$  is extremely small. Then every term in Eq. 12 and Eq. 13 will be a very large number, and as a result the extremely small number  $\xi$  will be of no importance in determining the evolution of  $a(t)$ . Therefore these two equations will be simplified to

$$\left(\frac{\ddot{a}}{a}\right)^2 - \left(\frac{\dot{a}}{a}\right)^4 = \left(\frac{A}{a}\right)^4, \quad (14)$$

$$2\left(\frac{\dot{a}}{a}\right)\left(-\frac{\ddot{a}}{a}\right) + \frac{d}{dt}\left(-\frac{\ddot{a}}{a}\right) + 2\left(\frac{\dot{a}}{a}\right)^3 = 0. \quad (15)$$

Eq. 14 and Eq. 15 will then describe the evolution of the early Universe.

It is interesting to point out that Eq. 14 and Eq. 15 will admit a simultaneous solution of the form of

$$a(t) = a_0(\cosh 2\beta t)^{\frac{1}{2}}, \quad (16)$$

where  $a_0$  and  $\beta$  are two integration constants related to the other integration constant  $A$  given in Eq. 11 by

$$A^4 = 4\beta^4 a_0^4. \quad (17)$$

The  $a_0$  here is the initial value of  $a(t)$  at  $t = 0$ , and when we compare Eq. 11 and Eq. 17, we will see immediately that the initial value of the energy density of the Universe at  $t = 0$  is related to  $\beta$  through

$$\rho(t = 0) = 24\kappa\beta^4. \quad (18)$$

If Eq. 16 is to describe the evolution of the Universe starting at  $t = 0$ , and if  $\beta$  is *an extremely large number*, say  $\beta = \frac{1}{4} \times 10^{35} \text{sec}^{-1}$ , and if the initial value of  $a(t)$  at  $t = 0$  (which is  $a_0$ ) is extremely small, then it is evident that the Universe will undergo an inflationary phase at the very beginning of its evolution. When the Universe evolves from the age of  $10^{-36}$  sec to  $10^{-33}$  sec, its radius will increase by a factor

$$\exp[2\beta(10^{-33}\text{sec})] / \exp[2\beta(10^{-36}\text{sec})] = 10^{23}. \quad (19)$$

#### IV. THE EVOLUTION OF THE LATE-TIME UNIVERSE

However the Universe won't keep on expanding forever like what is given by Eq. 16 because the simplified Eq. 14 and Eq. 15 will no longer describe the Universe when  $a(t)$  is

no longer extremely small. At the time the Universe is getting large enough, the right-hand-side of the cosmic equations of Eq. 12 will no longer be very large. Instead it gets very small very fast as time goes by, and at a point when the importance of  $\xi$  cannot be neglected, the simplified equations of Eq. 14 and Eq. 15 will no longer describe the Universe. What are describing the Universe now are the cosmic equations of Eq. 12 and Eq. 13 with their right-hand-sides vanishing. Eq. 12 and Eq. 13 with vanishing right-hand-sides have a trivial simultaneous solution of

$$\frac{\dot{a}}{a} - \xi = 0, \quad (20)$$

because

$$\left[ \frac{\ddot{a}}{a} - \xi \left( \frac{\dot{a}}{a} \right) \right] = \left[ \left( \frac{\dot{a}}{a} \right) \left( \frac{\dot{a}}{a} - \xi \right) + \frac{d}{dt} \left( \frac{\dot{a}}{a} - \xi \right) \right], \quad (21)$$

for a constant  $\xi$ .

The solution for Eq. 20 is

$$a = r_0 \exp(\xi t), \quad (22)$$

with  $r_0$  as an integration constant. In this solution the  $\rho$  and  $p$  are all equal to zero and hence satisfy  $p = \frac{1}{3}\rho$ .

## V. CONCLUSIONS

According to the  $GL(4, R)$  Yang-Mills theory of gravity, our Universe would inflate like  $a(t) = a_0(\cosh 2\beta t)^{\frac{1}{2}}$  if it has a very huge matter density  $\rho_0 = 24\kappa\beta^4$  at the very beginning of time. And then it would slow down and will take a much slower rate of expansion of  $a(t) = r_0 \exp(\xi t)$  in its late time evolution, if a torsion of extremely small local value  $\xi$  was created at the birth of the Universe. This small primordial torsion, however, has no effect on the evolution of the early Universe.

## VI. DISCUSSIONS

Even though we have carried out most of our discussions by using geometric languages such as the connections and the curvature tensor, these geometric objects are, in fact, derived from the gauge ideas of the gauge potentials and the Yang-Mills field strength tensor. The cosmic metric is driven in accordance with the evolution of an underlying gauge field



configuration because Eq. 8 is the Yang-Mills Equation for the  $GL(4\ R)$  group while Eq. 7 is the algebraic equations for the various components of the Yang-Mills strength tensor.

We have shown that the Yang-Mills gauge theory of gravity basing on the  $GL(4\ R)$  local group admits solutions which can give good descriptions of the early-time and late-time evolution of the Universe. No cosmological constant and no rolling scalar field is required. Because the solution given in Eq. 16 can be extrapolated back in time, it will take an infinite time for the Universe to evolve to any finite size if it starts at a state with zero size. So according to the  $GL(4\ R)$  Yang-Mills gauge theory of gravity, our Universe has existed in the infinite past, and takes an infinite time to evolve to our present day status, on the condition that has started at zero size and that we can ignore all the quantum effects during its evolution.

A solution of the form of  $a(t) = a_0(\cos 2\beta t)^{\frac{1}{2}}$  is also possible for our cosmic equations. This solution doesn't seem to give a proper description of our Universe, though.

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